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Cantor family of superstable manifolds of a double root in the dynamics of Newton's method ^{*†}

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Abstract

In the local dynamics of Newton's method, a generic double root of a holomorphic function of two variables has a Cantor family of holomorphic superstable manifolds.

1 Introduction

The aim of this paper is to give a geometric description on the local convergence of Newton's method toward a generic multiple root z_0 , in the case of a holomorphic function of two variables.

Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a holomorphic function defined locally on a neighborhood of a point z_0 . Newton's method of F is the mapping $NF(z) = z - (DF)_z^{-1}F(z)$ where $z = (x, y) \in \mathbb{C}^2$. If $L : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a linear automorphism, then we have $N(L \circ F) = NF$ and $N(F \circ L) = L^{-1} \circ NF \circ L$. The point z_0 is called a multiple root of F if $F(z_0) = (0, 0)$ and $\det(DF)_{z_0} = 0$.

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Suppose that $z_0 = (0, 0)$ is a ‘non-degenerate’ multiple root, that is, F is written, after a linear coordinate change, by

$$F(z) = (x + a_2x^2 + a_1xy + a_0y^2 + O(\|z\|^3), y^2 - x^2 + O(\|z\|^3)) \quad (1)$$

where $\|z\| = \max(|x|, |y|)$ is the box norm. Suppose furthermore that

$$a_2 + a_0 \neq \pm a_1. \quad (2)$$

We are going to show the followings. There exists a neighborhood $\mathbf{K} \ni z_0$ that is divided into three subsets

$$\mathbf{K} \setminus \{z_0\} = A \cup B \cup C \quad (3)$$

where

- A is called an attracting set. $NF(A) \subset A$. For each $z \in A$, we have $\|(NF)^{n+1}(z)\| / \|(NF)^n(z)\| \rightarrow 1/2$ as $n \rightarrow \infty$.
- B is called a bursting set. $B = \bigcup_{n=0}^{\infty} B_n$ where $B_0 = (NF|_{\mathbf{K}})^{-1}(\mathbf{C}^2 \setminus \mathbf{K})$, $B_n = (NF|_{\mathbf{K}})^{-n}(B_0)$. The image $(NF)^{n+1}(B_n)$ is unbounded. Each B_n consists of 2^n components.
- C is called a chaotic set, or a Cantor family of holomorphic superstable manifolds. There exist constants $0 < c_1 < c_2$ such that $c_1 |x|^2 \leq \|NF(z)\| \leq c_2 |x|^2$ for each $z \in C$.

Section 2 gives the decomposition (3). A keypoint is that the multiple root z_0 of F is an indeterminate point of NF . By choosing appropriate coordinates, we find a local blow-up transformation that is defined on a pair of polydiscs and is mapped to an unbounded region transversing themselves. Section 3 studies such a mapping, which we call a critical ‘dango’ (or ‘barbecue’) transformation.

By the C^r center manifold theorem (see [2]), we know that there exists a C^r invariant manifold of z_0 in the attracting set A , but its analyticity is not known. In section 4 we consider this problem in a general situation.

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2 A multiple root in Newton's method

Here we give the decomposition (3).

Newton's method of F as in (1) is written by

$$(X, Y) = NF(z) = \left(\frac{h_1(z)}{2y + h_0(z)}, \frac{y^2 - x^2 + h_2(z)}{2y + h_0(z)} \right) \quad (4)$$

where $h_0 = O(\|z\|^2)$, $h_1 = a_1x^3 + 2(a_2 + a_0)x^2y + a_1xy^2 + O(\|z\|^4)$, and $h_2 = O(\|z\|^3)$. Let c_{20} be the coefficient of x^2 in h_0 .

Given $0 < \varepsilon < 1$, let $A_0 = \{|x| < \varepsilon |y|\}$, $B'_0 = \{|y| < \varepsilon |x|\}$, and $C_0 = C_0^+ \cup C_0^- = \{|y - x| < \varepsilon |x|\} \cup \{|y + x| < \varepsilon |x|\}$. Given $\varepsilon' > 0$ and $0 < \delta < \varepsilon^2/4$, there exists a $0 < \rho \leq 1/(3\varepsilon')$ such that (i) $|h_0| < \delta \|z\|$, $|h_1| < \delta \|z\|^2$, and $|h_2| < \delta \|z\|^2$ in $\mathbf{K} = \{(x, y) \in \mathbf{C}^2 \mid |x| < \delta\rho, |y| < \rho\}$, and (ii) $|y^2 + h_2| < \frac{1}{3}|x|^2$ and $|h_0 - c_{20}x^2| < \varepsilon'|x|^2$ in $B''_0 = \{(x, y) \in \mathbf{K} \mid |2y + c_{20}x^2| < \varepsilon'|x|^2\}$.

Lemma 1 $B''_0 \subset B_0 \subset B'_0 \cap \mathbf{K}$.

(proof) If $(x, y) \in \mathbf{K} \setminus B'_0$ we have

$$|X| = \left| \frac{h_1}{2y + h_0} \right| < \frac{\delta\rho^2}{2\rho - \delta\rho} < \delta\rho,$$

$$|Y| = \left| \frac{y^2 - x^2 + h_2}{2y + h_0} \right| < \frac{\rho^2 + \delta^2\rho^2 + \delta\rho^2}{2\rho - \delta\rho} < \rho.$$

Thus $NF(x, y) \in \mathbf{K}$. For $(x, y) \in B''_0$, we have

$$|Y| > \frac{|x|^2 - \frac{1}{3}|x|^2}{\varepsilon'|x|^2 + \varepsilon'|x|^2} \geq \rho.$$

(qed)

Lemma 2 $NF(\mathbf{K} \setminus C_0) \subset A_0$.

(proof) If $(x, y) \notin C_0$, we have $|y^2 - x^2| > \frac{1}{2}\varepsilon \|z\|^2$ and

$$\left| \frac{X}{Y} \right| = \left| \frac{h_1}{y^2 - x^2 + h_2} \right| \leq \frac{\delta \|z\|^2}{\frac{1}{2}\varepsilon \|z\|^2 - \delta \|z\|^2} < \varepsilon.$$

(qed)

By this Lemma, $B_n \subset C_0$ for $n \geq 1$. Define $C = \bigcap_{n=0}^{\infty} C_n$, $C_n = (NF|_{\mathbf{K}})^{-n}(C_0)$, and $A = N \setminus (B \cup C) = \bigcup_{n=0}^{\infty} (NF|_{\mathbf{K}})^{-n}(A_0)$.

In the following three subsections, we describe the sub-dynamics in A_0 , B_0 , and C_0 .

2.1 Attracting set

Here we consider the dynamics in A_0 . Let $(x, y) = \phi(u, v) = (uv, v)$, $(U, V) = (U_1, V_1)$ where $(U_n, V_n) = (\phi^{-1} \circ NF \circ \phi)^n(u, v)$. Both U and V are divisible by v , and $(U/v, V/v)|_{(u,v)=(0,0)} = (0, 1/2)$. Thus by the standard argument similar to Schröder's equation (see [1]), $\varphi = \varphi(u, v) = \lim_{n \rightarrow \infty} 2^n V_n = v + \dots$ is uniformly convergent in a neighborhood of $(u, v) = (0, 0)$. Since φ/v is holomorphic around the origin $(u, v) = (0, 0)$, U is divisible by φ , and $\psi = U/\varphi$ is also holomorphic. By the new coordinates $(\xi, \eta) = (u, \varphi)$, we obtain the dynamics

$$(\xi, \eta) \mapsto (\eta\psi(\xi, \eta), \eta/2). \quad (5)$$

By the C^r center manifold theorem (see [2], Appendix III), we know that there exists a C^r differentiable function $\xi = \sigma(\eta) = \sigma(\text{re}(\eta), \text{im}(\eta))$ around the origin, whose graph is invariant under the dynamics (5). In section 4, we consider the problem whether this invariant manifold is holomorphic, in a general context.

2.2 Bursting set

Lemma 3 *The image $NF(B_0'') \subset NF(B_0)$ is unbounded.*

(proof) Given any $0 < \varepsilon'' < \varepsilon'$, take a point $z \in B_0''$ such that $|2y + c_{20}x^2| < \varepsilon''|x|^2$ and $|h_0 - c_{20}x^2| < \varepsilon''|x|^2$. Then we have

$$|Y| \geq \frac{|x|^2 - \frac{1}{3}|x|^2}{\varepsilon''|x|^2 + \varepsilon''|x|^2} = \frac{1}{3\varepsilon''}.$$

(qed)

As a description by coordinate geometry, let $(u, v) = (x, y/x^2)$ and $(\tilde{X}, \tilde{Z}) = (X/Y, 1/Y)$. Then $(u, v) = (0, v)$ is mapped to $(\tilde{X}, \tilde{Z}) = (0, -2v - c_{20})$. If $a_1 \neq 0$, this is a local diffeomorphism around each $(u, v) = (0, v)$.

2.3 Chaotic set

In (4), choose the coordinates $(u, v) = (x, y/x)$, $(U, V) = (X, X/Y)$. Let K_1 and K_2 be neighborhoods of $(u, v) = q_1 = (0, 1)$ and $q_2 = (0, -1)$ respectively. Let K be a neighborhood of the line $u = 0$. Around each q_i , the mapping $(u, v) \mapsto (\sqrt{U}, \sqrt{UV})$ is a local diffeomorphism with

$$\left. \frac{\partial(\sqrt{U}, \sqrt{UV})}{\partial(u, v)} \right|_{(u,v)=(0,\pm 1)} = \begin{pmatrix} \sqrt{\pm \frac{1}{2}(a_2 + a_0 \pm a_1)} & 0 \\ * & \sqrt{\pm 2(a_2 + a_0 \pm a_1)^{-1}} \end{pmatrix}$$

where \sqrt{U} is any branch. Thus we can apply Theorem 4, given in Section 3, to the local dynamics $K_1 \cup K_2 \rightarrow K$ to obtain the Cantor family of holomorphic curves $\sigma : \Sigma(2) \rightarrow \mathbf{H}_1 \cup \mathbf{H}_2$. By re-choosing δ sufficiently small if necessary, we obtain the chaotic set C as the graph $G(\sigma)$.

3 Cantor family of superstable manifolds

Here we give a prototype of a local dynamics that makes a Cantor family of holomorphic superstable manifolds. Let $i, j = 1, 2$ throughout this section.

Let $\pi(u, v) = (u, uv)$ and $\text{sq}(u, v) = (u^2, v)$ be mappings of \mathbf{C}^2 . Let K_0 be a neighborhood of the origin in \mathbf{C}^2 , and let $K = \pi^{-1}(K_0)$. Consider two points $q_i = (0, \alpha_i)$ and their neighborhoods $K_i \ni q_i$. Let $g_i : K_0 \rightarrow K_i$, $g_i(0, 0) = q_i$, be a biholomorphic map with its linear part $S_i(u, v) = (a_i u + b_i v, \alpha_i + c_i u + d_i v)$. We consider the local dynamics

$$f : K_1 \cup K_2 \rightarrow K, \quad \text{where } f|_{K_i} = \text{sq} \circ \pi^{-1} \circ g_i^{-1}.$$

(Note that the dynamics of a mapping like $\pi^{-1} \circ g_i^{-1} : K_i \rightarrow K$ was studied in [3].)

Let $\mathbf{B}_0 = \overline{\mathbf{D}}(0, \rho) \times \overline{\mathbf{D}}(0, r_0) \subset \overline{\mathbf{D}}(0, \sqrt{\rho}) \times \overline{\mathbf{D}}(0, r_0) \subset K_0$ be closed polydiscs where $0 < \rho < 1$ and $\mathbf{B}_i = \overline{\mathbf{D}}(0, \rho) \times \overline{\mathbf{D}}(\alpha_i, r) \subset K_i$. Let $\mathbf{L}_i = \text{Lip}_M(\overline{\mathbf{D}}(0, \rho), \overline{\mathbf{D}}(\alpha_i, r))$ be the set of Lipschitz functions of $\overline{\mathbf{D}}(0, \rho)$ to $\overline{\mathbf{D}}(\alpha_i, r)$ with Lipschitz constant $\leq M$, and its subset

$$\mathbf{H}_i = \left\{ \tau_i \in \mathbf{L}_i \mid \tau_i|_{\mathbf{D}(0, \rho)} \text{ is holomorphic} \right\}.$$

Let $\Sigma(2) = \{1, 2\}^{\mathbf{N}} \ni w = w_0 w_1 \cdots$ be a Cantor set.

Theorem 4 Suppose that $|a_i + b_i \alpha_j| \neq 0$, $i, j = 1, 2$. There exist $r, r_0, M > 0$, $0 < \rho < 1$, and a unique embedding (homeomorphism onto its image) $\sigma : \Sigma(2) \rightarrow \mathbf{H}_1 \cup \mathbf{H}_2$ such that

1. $\text{graph}(\sigma(w)) \cap \text{graph}(\sigma(w')) = \{q_{w_0}\}$ for any $w, w' \in \Sigma(2)$ with $w_0 = w'_0$.
2. σ is invariant under f : $\text{graph}(\sigma(w)) = \mathbf{B}_{w_0} \cap f^{-1}(\text{graph}(\sigma(s(w))))$ for each $w \in \Sigma(2)$.
3. The graph $G(\sigma) = \bigcup_{w \in \Sigma(2)} \text{graph}(\sigma(w))$ is the maximal local invariant set in $\mathbf{B}_1 \cup \mathbf{B}_2$: $G(\sigma) = \bigcap_{n=0}^{\infty} f^{-n}(\mathbf{B}_1 \cup \mathbf{B}_2)$.
4. $G(\sigma)$ is the local stable set of $\{q_1, q_2\}$, written by $W_{\text{loc}}^s(\{q_1, q_2\})$: $f^n(z) \rightarrow \{q_1, q_2\}$ as $n \rightarrow \infty$ for each $z \in G(\sigma) \setminus \{q_1, q_2\}$.
5. $G(\sigma)$ is the local ‘superstable’ set of $\{q_1, q_2\}$: there exist constants $0 < c_1 < c_2$ such that $c_1 |x|^2 \leq |p_1 f(z)| \leq c_2 |x|^2$ for each $z = (x, y) \in G(\sigma) \setminus \{q_1, q_2\}$.

The remainder of this section is a proof of this theorem.

Let $b = \max(|b_1|, |b_2|, |d_1|, |d_2|)$. Given $r > 0$ and $M > \left| \frac{c_i + d_i \alpha_j}{a_i + b_i \alpha_j} \right|$, there exist $r_0 > 0$ and $0 < \rho < 1$ that satisfy the followings: $\sqrt{\rho}(|\alpha_i| + r) \leq r_0$, $\rho M \leq r$, $\delta + \sqrt{\rho} \leq |a_i + b_i \alpha_j|$, $\frac{|c_i + d_i \alpha_j| + \delta}{|a_i + b_i \alpha_j| - \delta} \leq M$, $\delta_2 = (\ell + b)\sqrt{\rho}(1 + M) < 1$ where $\ell = \text{Lip}(g_i - S_i)$ is the Lipschitz constant as a mapping of $\overline{\mathbf{D}}(0, \sqrt{\rho}) \times \overline{\mathbf{D}}(0, r_0)$ and $\delta = \ell \max(1, |\alpha_j| + r + 2\rho^2 M) + b(r + 2\rho^2 M)$.

Denote by $\tau_j^*(u) = \pi(u, \tau_j(u^2))$ for $\tau_j \in \mathbf{L}_j$. We are going to define the graph transform

$$\Gamma_{g_i}(\tau_j) = p_2 g_i \tau_j^* (p_1 g_i \tau_j^*)^{-1}.$$

Lemma 5 $\Gamma_{g_i}(\tau_j) : \overline{\mathbf{D}}(0, \rho) \rightarrow \mathbf{C}$ is well-defined.

(proof) As a function of $\overline{\mathbf{D}}(0, \sqrt{\rho})$, we have

$$\text{Lip}(u \mapsto u(\tau_j(u^2) - \alpha_j)) \leq r + 2\rho^2 M.$$

Let $\tau_{j0} \in \mathbf{L}_j$ be the constant function $\tau_{j0}(u) = \alpha_j$, $k = 1, 2$. Then, as a function of $\overline{\mathbf{D}}(0, \sqrt{\rho})$, we have $\text{Lip}(\tau_j^*) \leq \max(1, |\alpha_j| + r + 2\rho^2 M)$, $\text{Lip}(p_k S_i \tau_j^* -$

$p_k S_i \tau_{j0}^*) \leq b(r + 2\rho^2 M)$, and

$$\begin{aligned} & \text{Lip}(p_k g_i \tau_j^* - p_k S_i \tau_{j0}^*) \\ & \leq \text{Lip}(p_k g_i \tau_j^* - p_k S_i \tau_j^*) + \text{Lip}(p_k S_i \tau_j^* - p_k S_i \tau_{j0}^*) \\ & = \delta. \end{aligned}$$

Since $p_1 S_i \tau_{j0}^*(u) = (a_i + b_i \alpha_j)u$ is a linear mapping with $|a_i + b_i \alpha_j| > \delta$, the Lipschitz Inverse Function Theorem ([2], Appendix I) can be applied. The mapping $p_1 g_i \tau_j^*$ is a homeomorphism of $\overline{\mathbf{D}}(0, \sqrt{\rho})$ onto its image, with

$$\text{Lip}([p_1 g_i \tau_j^*]^{-1}) \leq (|a_i + b_i \alpha_j| - \delta)^{-1}.$$

Thus the image contains $\overline{\mathbf{D}}(0, \sqrt{\rho}(|a_i + b_i \alpha_j| - \delta)) \supset \overline{\mathbf{D}}(0, \rho)$. (qed)

Lemma 6 $\Gamma_{g_i} : \mathbf{L}_1 \cup \mathbf{L}_2 \rightarrow \mathbf{L}_i$ is well-defined.

(proof) As a mapping on $\overline{\mathbf{D}}(0, \rho)$, we have

$$\begin{aligned} & \text{Lip}([p_1 g_i \tau_j^*]^{-1} - [p_1 S_i \tau_{j0}^*]^{-1}) \\ & \leq \text{Lip}([p_1 g_i \tau_j^*]^{-1}) \text{Lip}(p_1 g_i \tau_j^* - p_1 S_i \tau_{j0}^*) \text{Lip}([p_1 S_i \tau_{j0}^*]^{-1}) \\ & \leq \frac{\delta}{(|a_i + b_i \alpha_j| - \delta) |a_i + b_i \alpha_j|}. \end{aligned}$$

Then

$$\begin{aligned} \text{Lip}(\Gamma_{g_i}(\tau_j) - \Gamma_{S_i}(\tau_{j0})) & \leq \text{Lip}(p_2 g_i \tau_j^* - p_2 S_i \tau_{j0}^*) \text{Lip}([p_1 g_i \tau_j^*]^{-1}) \\ & \quad + \text{Lip}(p_2 S_i \tau_{j0}^*) \text{Lip}([p_1 g_i \tau_j^*]^{-1} - [p_1 S_i \tau_{j0}^*]^{-1}) \\ & \leq \frac{\delta}{|a_i + b_i \alpha_j| - \delta} \left(1 + \left| \frac{c_i + d_i \alpha_j}{a_i + b_i \alpha_j} \right| \right). \end{aligned}$$

Since $\Gamma_{S_i}(\tau_{j0})(u) = \alpha_j + (c_i + d_i \alpha_j)(a_i + b_i \alpha_j)^{-1}u$, we have

$$\text{Lip}(\Gamma_{g_i}(\tau_j)) \leq \text{Lip}(\Gamma_{g_i}(\tau_j) - \Gamma_{S_i}(\tau_{j0})) + \text{Lip}(\Gamma_{S_i}(\tau_{j0})) \leq M.$$

We also have $\Gamma_{g_i}(\tau_j)(0) = \alpha_i$ and $\rho M \leq r$, so $\Gamma_{g_i}(\tau_j) \in \mathbf{L}_i$. (qed)

Lemma 7 $\Gamma_{g_i} : \mathbf{L}_1 \cup \mathbf{L}_2 \rightarrow \mathbf{L}_i$ is a contraction:

$$\|\Gamma_{g_i}(\tau'_j) - \Gamma_{g_i}(\tau_j)\| \leq \delta_2 \|\tau'_j - \tau_j\|, \quad \tau_j, \tau'_j \in \mathbf{L}_j,$$

where $\|\cdot\|$ denotes the sup norm of a function on $\overline{\mathbf{D}}(0, \rho)$.

(proof) For a point $(u, v) \in \overline{\mathbf{D}}(0, \sqrt{\rho}) \times \overline{\mathbf{D}}(\alpha_i, r)$ we have

$$\begin{aligned} & \left| p_k g_i \pi(u, v) - p_k g_i \pi(u, \tau_j(u^2)) \right| \\ & \leq \text{Lip}(p_k) \text{Lip}(g_i - S_i) \left| \pi(u, v) - \pi(u, \tau_j(u^2)) \right| \\ & \quad + \left| p_k S_i \pi(u, v) - p_k S_i \pi(u, \tau_j(u^2)) \right| \\ & \leq (\ell + b) \sqrt{\rho} \left| v - \tau_j(u^2) \right|. \end{aligned}$$

Since $p_2 g_i \pi(u, \tau_j(u^2)) = \Gamma_{g_i}(\tau_j)(p_1 g_i \pi(u, \tau_j(u^2)))$ we obtain

$$\begin{aligned} & \left| p_2 g_i \pi(u, v) - \Gamma_{g_i}(\tau_j)(p_1 g_i \pi(u, v)) \right| \\ & \leq \left| p_2 g_i \pi(u, v) - p_2 g_i \pi(u, \tau_j(u^2)) \right| \\ & \quad + \text{Lip}(\Gamma_{g_i}(\tau_j)) \left| p_1 g_i \pi(u, \tau_j(u^2)) - p_1 g_i \pi(u, v) \right| \\ & \leq \delta_2 \left| v - \tau_j(u^2) \right|. \end{aligned}$$

Let $v = \tau'_j(u^2)$ and $u' = p_1 g_i \pi(u, \tau'_j(u^2))$ to obtain

$$\left| \Gamma_{g_i}(\tau'_j)(u') - \Gamma_{g_i}(\tau_j)(u') \right| \leq \delta_2 \left| \tau'_j(u^2) - \tau_j(u^2) \right|.$$

If u^2 runs in $\overline{\mathbf{D}}(0, \rho)$, u' runs in a region containing $\overline{\mathbf{D}}(0, \rho)$. (qed)

Two contraction mappings $\Gamma_{g_i} : \mathbf{L}_1 \cup \mathbf{L}_2 \rightarrow \mathbf{L}_i$ makes a homeomorphism (onto its image) $\sigma : \Sigma(2) \rightarrow \mathbf{L}_1 \cup \mathbf{L}_2$ by defining

$$\sigma(w) = \bigcap_{n=1}^{\infty} \Gamma_{g_{w_0}} \cdots \Gamma_{g_{w_{n-1}}}(\mathbf{L}_{w_n}).$$

Since $\Gamma_{g_i}(\mathbf{H}_1 \cup \mathbf{H}_2) \subset \mathbf{H}_i$, we have $\sigma(\Sigma(2)) \subset \mathbf{H}_1 \cup \mathbf{H}_2$. All the properties 1–5 are now clear from the construction of σ .

4 Invariant curve in the attracting set

In this section we consider the local dynamics $z = (x, y) \mapsto F(z) = (yf(z), \lambda y)$ where $f(0) = 0$ and $0 < |\lambda| < 1$, defined in a neighborhood of the origin. Our problem is the existence of a local holomorphic curve $x = \sigma(y)$ passing through the origin, forward invariant under F . If there exists such a $x = \sigma(y) = \sum_{n=1}^{\infty} c_n y^n$, then it has to satisfy the functional equation

$$yf(\sigma(y), y) = \sigma(\lambda y) \tag{6}$$

so that the coefficients c_n are uniquely determined.

Proposition 8 *If $f(z) = ax + by$ is a linear function with $b \neq 0$, there exists no invariant holomorphic curve $x = \sigma(y)$ that passes through the origin.*

(proof) From (6), we obtain $c_1 = 0$, $c_2\lambda = b$ and $c_{n+1}\lambda^n = ac_n$, $n \geq 2$. Thus $c_n = \lambda^{-n(n-1)/2}a^{n-2}b$, and the radius of convergence of σ is equal to 0. (qed)

Proposition 9 *For any holomorphic function $\sigma(y) = \sum_{n=2}^{\infty} c_n y^n$ there exists an f such that the graph $x = \sigma(y)$ is invariant under F .*

(proof) $f(x, y) = x - \sigma(y) + \sigma(\lambda y)/y$ for instance. (qed)

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